

Quasi-concave functions on antimatroids

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Abstract

In this paper we consider quasi-concave set functions defined on antimatroids. There are many equivalent axiomatizations of antimatroids, that may be separated into two categories: antimatroids defined as set systems and antimatroids defined as languages. An algorithmic characterization of antimatroids, that considers them as set systems, was given in [4]. This characterization is based on the idea of optimization using set functions defined as minimum values of linkages between a set and the elements from the set complement. Such set functions are quasi-concave. Their behavior on antimatroids was studied in [5], where they were applied to constraint clustering. In this work we investigate a duality between quasi-concave set functions and linkage functions. Our main finding is that an arbitrary quasi-concave set function on antimatroid may be represented as minimum values of some monotone linkage function.

keywords: *antimatroid, quasi-concave set function, monotone linkage function.*

1 Introduction

Let E be a finite set. A *set system* over E is a pair (E, \mathcal{F}) , where $\mathcal{F} \subseteq 2^E$ is a family of subsets of E , called *feasible* sets. We will use $X \cup x$ for $X \cup \{x\}$, and $X - x$ for $X - \{x\}$.

Definition 1.1 *A non-empty set system (E, \mathcal{F}) is an antimatroid if*

- (A1) *for each non-empty $X \in \mathcal{F}$, there is an $x \in X$ such that $X - x \in \mathcal{F}$*
- (A2) *for all $X, Y \in \mathcal{F}$, and $X \not\subseteq Y$, there exist an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.*

Any set system satisfying (A1) is called *accessible*.

Definition 1.2 A set system (E, \mathcal{F}) has the interval property without upper bounds if for all $X, Y \in \mathcal{F}$ with $X \subseteq Y$ and for all $x \in E - Y$, $X \cup x \in \mathcal{F}$ implies $Y \cup x \in \mathcal{F}$.

There are some different antimatroid definitions:

Proposition 1.3 [1][3] For an accessible set system (E, \mathcal{F}) the following statements are equivalent:

- (i) (E, \mathcal{F}) is an antimatroid
- (ii) \mathcal{F} is closed under union
- (iii) (E, \mathcal{F}) satisfies the interval property without upper bounds.

For a set $X \in \mathcal{F}$, let $\Gamma(X) = \{x \in E - X : X \cup x \in \mathcal{F}\}$ be the set of *feasible continuations* of X . It is easy to see that an accessible set system (E, \mathcal{F}) satisfies the interval property without upper bounds if and only if for any $X, Y \in \mathcal{F}$, $X \subseteq Y$ implies $\Gamma(X) \cap (E - Y) \subseteq \Gamma(Y)$.

The maximal feasible subset of set $X \subseteq E$ is called a *basis* of X . Clearly, by (ii), there is only one basis for each set. It will be denoted by $\mathcal{B}(X)$.

Definition 1.4 For any $k \leq |E|$ the k -truncation of a set system (E, \mathcal{F}) is a new set system defined by

$$\mathcal{F}_k = \{X \in \mathcal{F} : |X| \leq k\}.$$

If (E, \mathcal{F}) is an antimatroid, then (E, \mathcal{F}_k) is the k -truncated antimatroid [2].

The *rank* of a set $X \subseteq E$ is defined as $\varrho(X) = \max\{|Y| : (Y \in \mathcal{F}) \wedge (Y \subseteq X)\}$, the rank of the set system (E, \mathcal{F}) is defined as $\varrho(\mathcal{F}) = \varrho(E)$. Notice, that every antimatroid (E, \mathcal{F}) is also a k -truncated antimatroid, where $k = \varrho(\mathcal{F})$.

A k -truncated antimatroid (E, \mathcal{F}) may not satisfy the interval property without upper bounds, but it does satisfy the following condition:

$$\text{if } X, Y \in \mathcal{F}_{k-1} \text{ and } X \subseteq Y, \text{ then } x \in E - Y, X \cup x \in \mathcal{F} \text{ imply } Y \cup x \in \mathcal{F}. \quad (1)$$

A set system (E, \mathcal{F}) has the k -truncated interval property without upper bounds if it satisfies (1).

Theorem 1.5 [4] An accessible set system (E, \mathcal{F}) of rank k is a k -truncated antimatroid if and only if it satisfies the k -truncated interval property without upper bounds.

In this paper we consider quasi-concave set functions on truncated antimatroids.

Definition 1.6 A set function F defined on a set system (E, \mathcal{F}) is quasi-concave if for each $X, Y \in \mathcal{F}$, and for any maximal feasible subset Z of $X \cap Y$

$$F(Z) \geq \min\{F(X), F(Y)\}. \quad (2)$$

Originally, these functions were considered [6] on the Boolean 2^E , where the inequality (2) turns into the following condition

$$\text{for each } X, Y \subset E, F(X \cap Y) \geq \min\{F(X), F(Y)\}.$$

For this case, the correspondence between quasi-concave set functions and monotone linkage functions were established in [7].

A function $\pi : E \times 2^E \rightarrow \mathbf{R}$ is called a *monotone linkage function* if

$$\text{for all } X, Y \subseteq E \text{ and } x \in E, X \subseteq Y \text{ implies } \pi(x, X) \geq \pi(x, Y). \quad (3)$$

Consider $F : 2^E \rightarrow \mathbf{R}$ defined for each $X \subset E$

$$F(X) = \min_{x \in E-X} \pi(x, X). \quad (4)$$

It was shown [6], that F is quasi-concave, and, moreover, for every quasi-concave function F there exists a monotone linkage function π , which determines F in accordance with (4).

In this work we extend these results to truncated antimatroids. The family of feasible sets \mathcal{F} of a truncated antimatroid (E, \mathcal{F}) forms a meet semilattice $L_{\mathcal{F}}$, with the lattice operation:

$$X \wedge Y = \mathcal{B}(X \cap Y).$$

Hence, for truncated antimatroids the inequality (2) is converted to the inequality

$$F(X \wedge Y) \geq \min\{F(X), F(Y)\}$$

for each $X, Y \in L_{\mathcal{F}}$.

2 Main results

The following theorem characterizes quasi-concave functions defined on k -truncated antimatroids. Note, that in fact, we consider the functions defined only on \mathcal{F}_{k-1} .

Theorem 2.1 *A set function F defined on a k -truncated antimatroid (E, \mathcal{F}) is quasi-concave if and only if there exist a monotone linkage function π such that for each $X \in \mathcal{F}_{k-1}$*

$$F(X) = \min_{x \in \Gamma(X)} \pi(x, X). \quad (5)$$

Proof. Let a set function F defined as a minimum of a monotone linkage function π . Note, that since for any antimatroid the operator Γ is not-empty for each $X \in \mathcal{F}_{k-1}$, the definition (5) is correct. To prove that the function F is quasi-concave on \mathcal{F}_{k-1} , first note that

$$\text{for each } X \subset E, \Gamma(\mathcal{B}(X)) \subseteq E - X, \quad (6)$$

which immediately follows from the definition of basis.

Since $F(X \wedge Y) = \min_{x \in \Gamma(X \wedge Y)} \pi(x, X \wedge Y)$ there is $x^* \in \Gamma(X \wedge Y)$ such that $F(X \wedge Y) = \pi(x^*, X \wedge Y)$. Then, by (6), $x^* \in E - (X \cap Y)$, i.e., either $x^* \in E - X$ or $x^* \in E - Y$. Without loss of generality, assume that $x^* \in E - X$. Thus $X \wedge Y \subseteq X$, and $x^* \in E - X$, and $x^* \in \Gamma(X \wedge Y)$, that accordingly to (1) implies $x^* \in \Gamma(X)$. Finally,

$$F(X \wedge Y) = \pi(x^*, X \wedge Y) \geq \pi(x^*, X) \geq \min_{x \in \Gamma(X)} \pi(x, X) = F(X) \geq \min\{F(X), F(Y)\}.$$

To extend this function to the whole truncated antimatroid (E, \mathcal{F}) we can define $F(X)$ for each maximal X , i.e., for $|X| = k$, as $F(X) = \min_{(x, X)} \pi(x, X)$. It is easy to check that this extension is quasi-concave too.

Conversely, let we have a quasi-concave set function F . Define the function

$$\pi_F(x, X) = \begin{cases} \max_{A \in [X, E-x]_{\mathcal{F}_{k-1}}} F(A), & x \notin X \text{ and } [X, E-x]_{\mathcal{F}_{k-1}} \neq \emptyset \\ \min_{A \in \mathcal{F}_{k-1}} F(A), & \text{otherwise} \end{cases} \quad (7)$$

The function π_F is monotone. Indeed, if $x \in E - Y$ and $[Y, E-x]_{\mathcal{F}_{k-1}} \neq \emptyset$, then $X \subseteq Y$ implies

$$\pi(x, X) = \max_{A \in [X, E-x]_{\mathcal{F}_{k-1}}} F(A) \geq \max_{A \in [Y, E-x]_{\mathcal{F}_{k-1}}} F(A) = \pi(x, Y).$$

It is easy to verify the remaining cases.

Let us denote $G(X) = \min_{x \in \Gamma(X)} \pi_F(x, X)$, and prove that $F = G$ on \mathcal{F}_{k-1} .

Now

$$G(X) = \min_{x \in \Gamma(X)} \pi_F(x, X) = \pi_F(x^*, X) = \max_{A \in [X, E-x^*]_{\mathcal{F}_{k-1}}} F(A) \geq F(X).$$

On the other hand,

$$G(X) = \min_{x \in \Gamma(X)} \pi_F(x, X) = \min_{x \in \Gamma(X)} F(A^x),$$

where A^x is a set from $[X, E-x]_{\mathcal{F}_{k-1}}$ on which the value of the function F is maximal, i.e.,

$$A^x = \arg \max_{A \in [X, E-x]_{\mathcal{F}_{k-1}}} F(A).$$

From quasi-concavity of F follows that

$$\min_{x \in \Gamma(X)} F(A^x) \leq F(\wedge_{x \in \Gamma(X)} A^x).$$

So, $G(X) \leq F(\wedge_{x \in \Gamma(X)} A^x)$.

It remains to prove, that for all $X \in \mathcal{F}_{k-1}$, $X = \wedge_{x \in \Gamma(X)} A^x$, where the set $A^x \in [X, E-x]_{\mathcal{F}_{k-1}}$.

Denote, $Y = \bigwedge_{x \in \Gamma(X)} A^x$. For each $x \in \Gamma(X)$, $X \subseteq A^x$, and consequently $X \subseteq Y$. Assume, that $X \subset Y$, then by definition (A2) there exists an element $y \in Y - X$ such that $X \cup y \in \mathcal{F}$, i.e., $y \in Y \cap \Gamma(X)$. On the other hand,

$$Y = \bigwedge_{x \in \Gamma(X)} A^x \subseteq \bigcap_{x \in \Gamma(X)} A^x \subseteq E - \Gamma(X).$$

This contradiction proves that $X = Y$.

Therefore, $G(X) \leq F(X)$, and, hence, $F = G$, i.e. $F(X) = \min_{x \in \Gamma(X)} \pi_F(x, X)$, where π_F is a monotone linkage function. ■

Thus, we proved that each quasi-concave function F determines a monotone linkage function π_F , and the set function defined as the minimum of this monotone linkage function π_F coincides with the original function F . A weaker property holds for the linkage functions.

Theorem 2.2 *Let $F(X) = \min_{x \in \Gamma(X)} \pi_F(x, X)$ for a monotone linkage function π on a k -truncated antimatroid (E, \mathcal{F}) . Then $\pi_F|_{\mathcal{F}_{k-1}} \leq \pi|_{\mathcal{F}_{k-1}}$, i.e., for any $X \in \mathcal{F}_{k-1}$ and $x \in \Gamma(X)$*

$$\pi_F(x, X) \leq \pi(x, X),$$

where π_F is defined by (7).

Proof. For any $X \in \mathcal{F}_{k-1}$ and $x \in \Gamma(X)$

$$\pi_F(x, X) = \max_{A \in [X, E-x]_{\mathcal{F}_{k-1}}} F(A) = F(A^*) = \min_{a \in \Gamma(A^*)} \pi(a, A^*) \leq \pi(x, A^*).$$

The last inequality follows from the k -truncated interval property without upper bounds. Indeed, $X \subseteq A^*$ and $x \notin A^*$, then $x \in \Gamma(X)$ implies $x \in \Gamma(A^*)$.

Now, from monotonicity of the function π we have $\pi(x, A^*) \leq \pi(x, X)$, that finishes the proof. ■

Consider the following example to see that these two functions π and π_F may be not equal. For example, let $E = \{1, 2\}$, $\mathcal{F} = 2^E$, and

$$\pi(x, X) = \begin{cases} 2, & x = 2 \text{ and } X = \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

Then the function $F(X) = \min_{x \in \Gamma(X)} \pi(x, X)$ is equal to 1 for all $X \subset E$, and π_F equals for 1 for each pair $(x, X) \in E \times 2^E$, i.e., $\pi_F \neq \pi$.

Now let us define more exactly the structure of the set of monotone linkage functions.

Theorem 2.3 *Let (E, \mathcal{F}) be a set system of rank k , where the set of feasible continuations of X is not empty for each $X \in \mathcal{F}_{k-1}$, and let π_1 and π_2 define (by (5)) the same set function F on \mathcal{F}_{k-1} . Then the function*

$$\pi = \min\{\pi_1, \pi_2\}$$

is a monotone linkage function, and it determines the same function F on \mathcal{F}_{k-1} .

Proof. At first, prove that π is a monotone linkage function. Indeed, consider a pair $X \subseteq Y$. Suppose, without loss of generality, that $\min\{\pi_1(x, X), \pi_2(x, X)\} = \pi_1(x, X)$. Now,

$$\begin{aligned}\pi(x, X) &= \min\{\pi_1(x, X), \pi_2(x, X)\} = \pi_1(x, X) \geq \\ &\geq \pi_1(x, Y) \geq \min\{\pi_1(x, Y), \pi_2(x, Y)\} = \pi(x, Y)\end{aligned}$$

To complete the proof, we show that

$$\begin{aligned}\min_{x \in \Gamma(X)} \pi(x, X) &= \pi(x^*, X) = \min\{\pi_1(x^*, X), \pi_2(x^*, X)\} \geq \\ &\geq \min\left(\min_{x \in \Gamma(X)} \pi_1(x, X), \min_{x \in \Gamma(X)} \pi_2(x, X)\right) = F(X),\end{aligned}$$

and on the other hand,

$$F(X) = \min_{x \in \Gamma(X)} \pi_1(x, X) = \pi_1(x^\#, X) \geq \pi(x^\#, X) \geq \min_{x \in \Gamma(X)} \pi(x, X).$$

■

Thus, the set of monotone linkage functions, defining a set function F on a truncated antimatroid, forms a semilattice with the following lattice operation:

$$\pi_1 \wedge \pi_2 = \min\{\pi_1, \pi_2\},$$

where by Theorem 2.2 the function π_F is a null of this semilattice.

The following theorem demonstrates the necessity of interval property for the above established correspondence between quasi-concave set functions and monotone linkage functions.

Theorem 2.4 *Let (E, \mathcal{F}) be an accessible set system of rank k . If the set of feasible continuations of X is not empty for each $X \in \mathcal{F}_{k-1}$, then the following statements are equivalent*

- (i) *(E, \mathcal{F}) is a k -truncated antimatroid*
- (ii) *the function $F = \min_{x \in \Gamma(X)} \pi(x, X)$ is quasi-concave for every monotone linkage function π .*

Proof. Since the one direction is proved (see Theorem 2.1), assume that the set system (E, \mathcal{F}) is not k -truncated antimatroid, i.e., there exist $A, B \in \mathcal{F}_{k-1}$ such that $A \subset B$, and there is $a \in E - B$ such that $A \cup a \in \mathcal{F}$ and $B \cup a \notin \mathcal{F}$. Define the linkage function

$$\pi(x, X) = \begin{cases} 0, & x \in X \\ 1, & x = a \text{ and } A \subseteq X \subseteq E - a \\ 2, & \text{otherwise} \end{cases}$$

It is easy to check that π is monotone.

Here, $F(A) = 1$, $F(A \cup a) = F(B) = 2$. Since $(A \cup a) \cap B = A$, we have

$$F((A \cup a) \cap B) < \min\{F(A \cup a), F(B)\},$$

i.e., F is not quasi-concave. ■

3 Conclusions

In this article, we discuss the duality between quasi-concave set functions and monotone linkage functions. It is shown that each quasi-concave function F , defined on an antimatroid, determines a semilattice of monotone linkage functions each of them defines the set function F , and the null of this semilattice is the function π_F constructed from the function F .

As the directions for future research we see the extension of the duality to other families of sets such as convex geometries, interval greedoids and more general set families.

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